MTH 310 HW 9 Solutions

April 8, 2016

Section 4.5, Problem 12

Prove that if $c \in \mathbb{F}$ and $f(x) \in \mathbb{F}[x]$ and $f(x+c)$ irreducible, then so is $f(x)$.
Answer. We prove by contrapositive. Assume f is reducible. Then $f(x) = p(x)q(x)$ for some $p(x), q(x) \in \mathbb{F}[x]$ with positive degree. Then $f(x+c) = p(x+c)q(x+c)$, and since $deg(p(x)) = deg(p(x+c))$ and $deg(q(x)) = deg(q(x+c))$, $f(x+c)$ is a product of polynomials of positive degree and thus is reducible.

Section 4.6, Problem 8

If $a + bi \in \mathbb{C}$ is a root of $p(x) = x^3 - 3x^2 + 2ix + i - 1$, must $a - bi$ also be?
Answer. No. One way to show this is to note that i is a root but $-i$ isn't.

But a more fun way (which might generalize a little better) goes as follows. Assume that if $p(a+bi) = 0$ then $p(a-bi) = 0$. We know by the fundamental theorem of algebra that there are exactly three roots of this polynomial. We claim that p has a real root. To show this, denote one of the roots as $r_1 = a + bi$ (guaranteed to exist by the fundamental theorem of algebra). If r_1 is real, we have shown what we are trying to show, so assume it isn't. Then $r_2 = a - bi$ must also be a root by our above assumption. But by corollary 4.28 (version 3), there is a third root of p–call it $r_3 = l + qi$. But then $l - qi$ must also be a root. But $l - qi$ it can't be r_1 or r_2 since that would mean r_3 was r_2 or r_1 respectively. Thus $r_3 = p - qi$ so $q = 0$ and p has a real root l.

Thus $0 = p(l) = l^3 - 3l^2 - 1 + i(2l+1)$, so $2l + 1 = 0$ and thus $l = -\frac{1}{2}$. But $p(l) \neq 0$ so there are no real roots.

1 Section 5.1, Problem 3

List the congruence classes in $\mathbb{Z}[x]\backslash(x^3 + x + 1)$.
Answer. Any congruence class has an element of degree 2 or less and if $p(x), q(x) \in \mathbb{Z}_2[x]$ with $[p(x)] = [q(x)]$ then $p(x) = q(x)$ so the classes are $[0], [1], [x], [x+1], [x^2], [x^2+1], [x^2+x], [x^2+x+1]$.